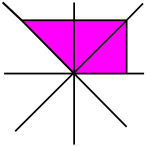
Signed Poset Polytopes



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What's to come....

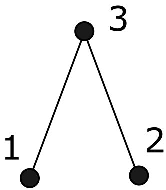
- Posets geometrically
- Generalizations of posets
  - Signed posets (Reiner, 1993)
  - Coxeter cones (Stembridge, 2007)
- Bringing things together: Signed order polytopes

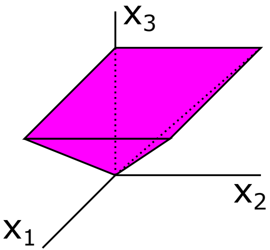
Order Polytopes

Let Π be a poset on n elements.

Order polytope — $\mathcal{O}(\Pi)$ of Π is the subset of $\mathbb{R}^{\Pi}$ given by:

$$\mathcal{O}(\Pi) = \{\phi \in \mathbb{R}^{\Pi} : 0 \leq \phi(p) \leq 1 \text{ for all } p \in \Pi \text{ and } \phi(a) \leq \phi(b) \text{ when } a \leq b\}$$



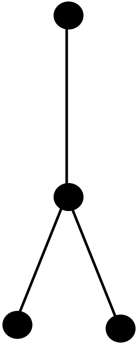


Order cone — $\mathcal{O}(\Pi)$ of Π is the subset of $\mathbb{R}^{\Pi}$ given by:

$$\mathcal{O}(\Pi) = \{\phi \in \mathbb{R}^{\Pi} : \phi(a) \leq \phi(b) \text{ when } a \leq b\}$$

From poset to order polytope (classical case)

- The dimension of $\mathcal{O}(\Pi)$ is the number of elements of Π.
- The vertices of $\mathcal{O}(\Pi)$ correspond to the filters of Π.
- The facets correspond to the maximal/ minimal elements and the cover relations of the poset.
- Volume can be computed from number of linear extensions.



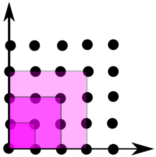
Why do we care about order polytopes?

- We know a lot about $\mathcal{O}(\Pi)$, but they can still be complicated.
  - Volume is computationally hard to compute.
- Potential place to look for examples of polytopes with nice properties.
  - Gorenstein ...
- An entry point to some open problems!
  - unimodality....

Ehrhart theory detour

Let P be a d-dimensional lattice polytope.

Ehrhart polynomial :
$\mathrm{ehr}_P(n) = |nP \cap \mathbb{Z}^d|$ This is indeed a polynomial in n. (Ehrhart 1962)



Ehrhart series:

$$Ehr_P(z) := 1 + \sum_{n \geq 1} ehr_P(n) z^n = \frac{h_0^* + h_1^* z + \cdots + h_{d+1}^* z^d}{(1-z)^{d+1}}$$

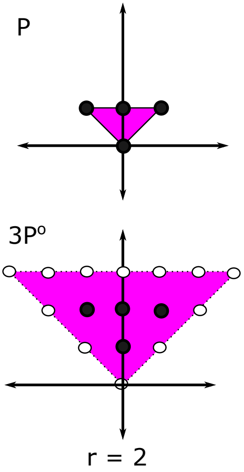
We call $(h_0^*, \ldots h_d^*)$ the h^*-vector of P.

Gorenstein polytopes

Let P be a d-dimensional lattice polytope and P° be its interior.

P is Gorenstein if and only if there exists a positive integer r such that:

- $(r-1)P^\circ \cap \mathbb{Z}^d = \emptyset$
- $|rP^\circ \cap \mathbb{Z}^d| = 1$
- $|tP^\circ \cap \mathbb{Z}^d| = |(t-r)P \cap \mathbb{Z}^d|$ for all integers $t > r$



Gorenstein order polytopes

- A lattice polytope is Gorenstein if and only if its h^*-polynomial has symmetric coefficients.
- The order polytope of a poset P is Gorenstein if and only if P is graded. (Hibi, Stanley)

Unimodality Questions

Unimodality: $a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_d$

Big Question: Do _____ have unimodal h^*-vectors?

Possible candidates for _____:

- IDP polytopes
- Polytopes that admit a unimodular triangulation
- _____
- Order polytopes

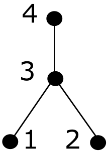
All of these questions are still open!

Signed Posets (Reiner, 1993)

Type A root system:

$$\{e_i - e_j : 1 \leq i \neq j \leq n \subset \mathbb{R}^n\}$$

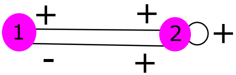
$$P = \{e_4 - e_3, e_3 - e_1, e_3 - e_2, e_4 - e_1, e_4 - e_2\}$$



Type B root system:

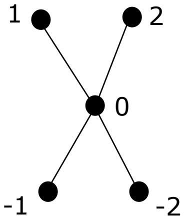
$$\{\pm e_i \pm e_j : 1 \leq i \neq j \leq n \subset \mathbb{R}^n\} \cup \{\pm e_i : 1 \leq i \leq n\}$$

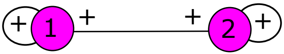
P = {e1+e2,-e1+e2,e2}



Fischer representation

Signed posets on $[n]$ can also be represented as classical posets on $\{-n, -(n-1), \ldots, 0, \ldots, n-1, n\}$. (Fischer, 93)





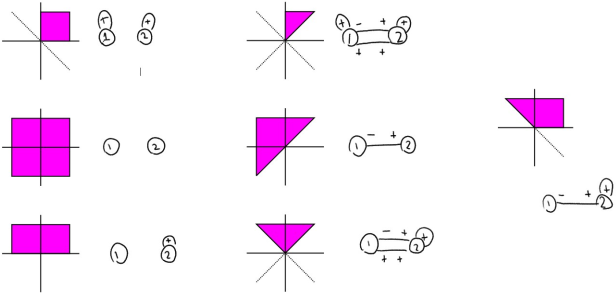
Coxeter Cones (Stembridge 2007)

- Introduced coxeter cones - formed from subsets of any root system.
- Determined exactly when the magentah- vectors of these cones are symmetric

Signed Order Polytopes

Definition

The *order polytope* of a signed poset P, $\mathcal{O}(P)$, is given by intersecting the cube $[-1,1]^n$ with the halfspaces given by $\langle \alpha, x \rangle \geq 0$ for all $\alpha \in P$.

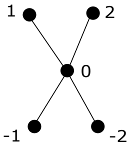


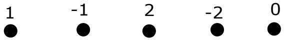
From signed poset to signed order polytope

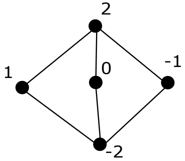
- If P is a signed poset on $[n]$, the dimension of $\mathcal{O}(P)$ is n
- $\mathcal{O}(P)$ can be written as the convex hull of the signed filters of P.
- The facets correspond to the signed maximal elements and the nonredundant relations of the poset.
- Volume can be computed from number of signed linear extensions.

Gorenstein example

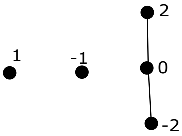
Gorenstein signed order polytopes

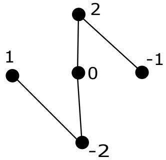






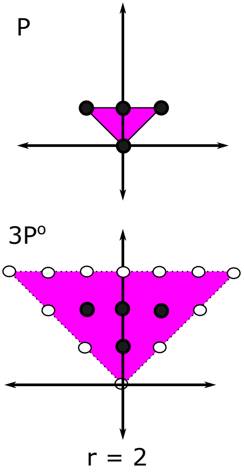
non-Gorenstein signed order polytopes





Gorenstein signed order polytopes

(Implied by Stembridge) Let P be a signed poset on $[n]$. Then $\mathcal{O}(P)$ is Gorenstein if and only if the Fischer poset representation of P is graded.



Unimodality

Theorem (Bruns, Roemer, 2005) A Gorenstein lattice polytope P with a regular unimodular triangulation has a unimodal h^*-vector.

Corollary: Let P be a signed poset on $[n]$. Then $\mathcal{O}(P)$ has unimodal h^*-polynomial if the Fischer poset representation of P is graded.

Further questions

- Stanley defined chain polytopes in 1986, what is the analogue here?

Thank you all! :)